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PhD THESIS
– ABSTRACT –

Fixed Structure Robust Synthesis for Nonlinear Systems

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1 Introduction

The current thesis constitutes fundamental research that aims to extend the mathematical framework to solve the fixed-structure robust synthesis control problem for nonlinear systems. The starting point is the seminal work [1], which states, based on several previous papers, that the $\mathcal{H}_\infty$ control problem is solved under the following assumptions: the process is linear and time-invariant, and the controller coefficients are expressed only in terms of arithmetic operations of the tunable parameters.

To design a suitable control structure, several subfields of the Control Engineering domain should be considered. First, an abstract mathematical model of the process should be obtained. This step is generally performed through modeling, system identification, or a combination of these two techniques. After an initial analysis of the system, a set of performances is imposed for an adequate control structure. The controller synthesis is performed such that the desired specifications are obtained. If the resulting controller is of high order, an additional order reduction step with minimum performance degradation must be made to prepare the second part of the flow. All these steps are generally performed in the continuous-time domain. The second part is the implementation of the obtained control structure, which should be performed in a computer-based environment. Even if the continuous-time controller is of a fixed structure or is a reduced-order representation of the initial controller, a discrete-time counterpart should be obtained. However, considering the sampling rate and the discretization method, infinitely many possible representations of such controllers exist. Moreover, considering the quantization effect applied to the controller's coefficients, input-output signals, and computations, a rigorous analysis of the performance degradation should be performed.

In the literature, all mentioned steps are treated individually, considering various criteria to perform the analysis of performance degradation. Additionally, the robustness of the fractional-order controllers is widely accepted in literature to be valid. However, as in the general case of the Mathematical Analysis field, the vast majority of phenomena and results seem to be obviously true, without having formal proof. But, in those cases, the results are often wrong, the idea of giving counterexamples in Mathematical Analysis being a continuous challenge. Motivated by this premise, in this thesis, I consider a unified approach to study the robustness of fractional-order controllers, using a unique metric to analyze the performance degradation at each step mentioned in the previous paragraph.

Because the fixed-structure robust synthesis control problem is considered solved for linear and time-invariant systems, a natural next step is to take a look at the same problem for nonlinear systems. Even if the problem is widely studied in the literature, the available solutions are made for particular classes of nonlinear systems, such as bilinear, input-affine, or polynomial. There are two main approaches to solving this problem: find a nonlinear mechanism that can be adapted for each subproblem individually, or find an intermediate mechanism that transforms a generic nonlinear system into a specific structure, such as linear or bilinear, which already has a solution for the fixed-structure robust synthesis problem or for which a solution could be developed. The thesis presents several attempts to solve the problem from the perspective of the second approach. One major direction is the possibility of developing fixed-structure passivity-based controllers for bilinear systems. Another major direction is to encompass all residual nonlinearities that appear when the mechanism to solve the exact feedback linearization problem is used for uncertain input-affine nonlinear systems.

The thesis contains four major parts. **Part I** presents the context and the main motivation, along with the objectives and the summary of the thesis. **Part II** contains two chapters and illustrates the current stage of knowledge, offering the mathematical background necessary for developing the main results. **Part III** is the central part of the thesis and contains three chapters, each one being a self-contained unit, containing a short literature review, along with research gaps and contributions, a detailed presentation of the main theoretical results, followed by a set of case studies to illustrate the relevance of the developed theory and a short conclusion of the chapter. **Part IV** deals with final discussions, followed by conclusions and further research directions. Additionally, the thesis presents the list of notations, abbreviations, tables, and figures at the beginning of the manuscript, while the bibliography and the list of publications close the thesis.

Part I: corresponds to the content of **Chapter 1**, presenting the scientific context, motivation, specific objectives, and a detailed overview of the thesis structure and contents.

2 Current Stage of Knowledge

Part II: contains **Chapters 2** and **3** as a mathematical background of the control methods developed in this thesis. **Chapter 2** contains a comprehensive overview of the available results to perform a robust control synthesis for a linear and time-invariant plant [2, 3]. For robust fixed-structure synthesis that minimizes the $\mathcal{H}_\infty$-norm, the paper [1] summarizes a series of results based on nonsmooth optimization techniques. To extend the robust control problem for uncertain systems, the structured singular value is used as a measure of robustness [4]. This leads to the μ-synthesis problem, which is NP-hard, being solved using the well-known D/G–K iteration [4], for which there is currently an approach based on nonsmooth optimization [5].

Chapter 3 presents the main tools available for analyzing the stability and performance of nonlinear systems, along with two controller synthesis methods: one using the passivity framework and one using the feedback linearization framework. The chapter begins with an overview of the main concepts for the stability of nonlinear systems [6]. After different types of stability concepts have been introduced, a brief presentation of results in the control field based on passivity theory follows [6]. This exposure focuses mainly on the recently defined concept of Krasovskii passivity [7], starting from a set of necessary and sufficient conditions to ensure Krasvoskii passivity for a nonlinear system and followed by designing Krasovskii passive controllers to ensure asymptotic stability. The chapter concludes with a brief overview of the main results in the field of state feedback linearization for input affine nonlinear systems, for both single-input single-output systems and multivariable systems [8].

3 Personal Contribution

Part III: contains **Chapters 4**, **5**, and **6** which encompass the main contributions of the thesis. **Chapter 4** presents the mechanism that allows to introduce the fractional-order element into the integer-order generalized Robust Control Framework by using the mathematical tools underlined in **Chapter 2**. **Chapters 5** and **6** deal with the possibility to solve the fixed-structure robust synthesis for nonlinear systems using Krasovskii passivity and feedback linearization, respectively, extending the notions briefly presented in **Chapter 3**.

Chapter 4 focuses on introducing the fractional-order element into the integer-order generalized Robust Control Framework. A finite order representation is necessary to implement such a controller, which is of infinite order. The proposed fractional-order controller family $\mathcal{K}$ of $n_y \times n_u$ fractional-order controllers can be written as:

$$\mathcal{K} = \left\{ \begin{pmatrix} H^{\mathbf{b}^{(1,1)},\beta^{(1,1)}}_{\mathbf{a}^{(1,1)},\alpha^{(1,1)}}(s) & \dots & H^{\mathbf{b}^{(1,n_u)},\beta^{(1,n_u)}}_{\mathbf{a}^{(1,n_u)},\alpha^{(1,n_u)}}(s) \\ \vdots & \ddots & \vdots \\ H^{\mathbf{b}^{(n_y,1)},\beta^{(n_y,1)}}_{\mathbf{a}^{(n_y,1)},\alpha^{(n_y,1)}}(s) & \dots & H^{\mathbf{b}^{(n_y,n_u)},\beta^{(n_y,n_u)}}_{\mathbf{a}^{(n_y,n_u)},\alpha^{(n_y,n_u)}}(s) \end{pmatrix} \right\}, \quad \text{where} \quad H^{\mathbf{b},\beta}_{\mathbf{a},\alpha}(s) = \frac{\sum\limits_{(b_i,\beta_i)\in\mathbf{b}\times\beta} b_i s^{\beta_i}}{\sum\limits_{(a_i,\alpha_i)\in\mathbf{a}\times\alpha} a_i s^{\alpha_i}}, \tag{1}$$

having the tunable parameters $\alpha^\top = (\alpha_1\ \alpha_2 \dots\ \alpha_n) \in \mathbb{R}^n$, $\mathbf{a}^\top = (a_1\ a_2 \dots\ a_n) \in \mathbb{R}^n$, $\beta^\top = (\beta_1\ \beta_2 \dots\ \beta_m) \in \mathbb{R}^m$, and $\mathbf{b}^\top = (b_1\ b_2 \dots\ b_m) \in \mathbb{R}^m$. Using both Oustaloup Recursive Approximation and Oustaloup Approximation, the fractional-order element is included into the fixed-structure robust synthesis considering an appropriate isomorphic representation, as underlined in Algorithms **3** and **4**. Additionally, the difference between the fractional-order element and its integer-order approximation is modelled using an additive uncertainty, presenting an uncertainty block $\mathbf{\Delta}_K$, as in Figure 1. Therefore, the fixed-structure μ-synthesis control problem must be converted into the following form:

$$\inf_{\overline{K}\in\widetilde{\mathcal{K}}_{IO}} \mu_{\text{diag}(\mathbf{\Delta},\mathbf{\Delta}_K)} \left(\text{LLFT}(P,\overline{K})\right), \tag{2}$$

where $\overline{K}$ is the controller approximation with an additional LLFT connection with the uncertainty block $\mathbf{\Delta}_K$, marked with dashed line in Figure 1, while $\widetilde{\mathcal{K}}_{\text{IO}}$ is the integer-order approximation of the fractional-order controller family $\mathcal{K}$. The same mechanism should be used to solve this modified version of the fixed-structure μ-synthesis control problem, the main advantage being that the fractional-order controller itself ensures robust stability and robust performance.

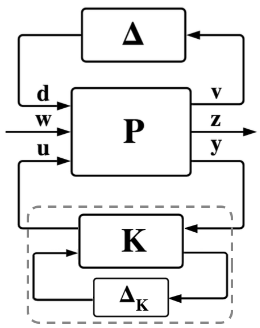

Figure 1: The generalized plant P with an ULFT connection with the uncertainty block $\mathbf{\Delta}$ and a LLFT connection with the controller $\overline{K}$, which consists in a LLFT connection between the controller K and the uncertainty block $\mathbf{\Delta}_K$ derived from the approximation of each fractional-order element of the controller K.

As stated in the literature review, there are several methodologies to design robust fractional-order controllers. However, from my findings, the thesis presents the first methodology to design a robust fractional-order controller that simultaneously guarantees the robustness of its integer-order approximation. A possible disadvantage of the method is the conservativism of the uncertainty model that encompasses the difference between the fractional-order element and its infinite impulse response approximation. This disadvantage may be limited if the integer-order approximation is of high order. But, in the design phase, the advantage of not being limited to fixed slopes of ±20[dB/dec] for the loop shaping procedure gives a finer tool and allows to impose better performance with fewer design parameters.

Another contribution of the chapter consists of the order reduction techniques which keep the robustness using the same metric as in the design phase. As such, for an integer-order approximation $\overline{K} \in \widetilde{\mathcal{K}}_{IO}$, the same quantity should be minimized in both cases:

$$\overline{\operatorname{ord}(K)} \equiv \sum_{i=1}^{n_y}\sum_{j=1}^{n_u}\sum_{\alpha_k\in\boldsymbol{\alpha}^{(i,j)}} N_{\alpha_k} + \sum_{i=1}^{n_y}\sum_{j=1}^{n_u}\sum_{\beta_k\in\boldsymbol{\beta}^{(i,j)}} N_{\beta_k}, \tag{3}$$

the resulting optimization problem being:

$$\begin{aligned} &\min \ \overline{\operatorname{ord}(K)} \\ &\text{s.t.} \ \ \mu_{\boldsymbol{\Delta}}\left(\operatorname{LLFT}(P, \overline{K})\right) < 1. \end{aligned} \tag{4}$$

Problem (4) can be seen as a topological sorting on a direct acyclic graph $G_{\mathbb{L}} = (V, E)$, where V are the vertices of the graph and E are the edges of the graph. The set V corresponds to the lattice's vertices $\mathbb{L} = \{1, 2, \ldots, N_0\}^{m+n} \subset \mathbb{Z}^{m+n}$, where N_0 is the initial order of approximation of each fractional-order element from $K \in \mathcal{K}$. The direct edges E are considered as follows: e_{ij} is considered between two vertices v_i and v_j if exactly one lattice coordinate varies by 1, while the rest remain the same, and both vertices produce an approximation which ensures robust stability and robust performance.

Lemma 1 *The optimal solution of the optimization problem* (4) *corresponds to the longest path of the direct acyclic graph* $G_{\mathbb{L}} = (V, E)$.

The second order reduction step is based on Hankel singular values. As proved in the second numerical example, each order reduction step must be performed, because the individual steps are not able to reduce the integer-order representation of the fractional-order controller enough. Additionally, the fitting mechanism is necessary, because each step causes a degradation, especially in terms of the phase. Based on a similarity functional, an optimization problem is obtained:

$$\begin{aligned} &\min_{K_{\text{approx}}} \sum_{\omega\in\overline{\Omega}} \|K(j\omega) - K_{\text{approx}}(j\omega)\|^2 \cdot \Delta\omega \\ &\text{s.t.} \quad \mu_{\boldsymbol{\Delta}}\left(\operatorname{LLFT}(P, K_{\text{approx}})\right) < 1. \end{aligned} \tag{5}$$

This fitting mechanism also takes the advantage of simultaneously having a fractional-order controller and its approximation with mathematical robustness guarantees, because the steps performed by the optimization algorithm can be performed only in the feasible region.

These order reduction problems, along with the problem of choosing the sampling rate and the uniform quantization step, solve the numerical implementation problem of a fractional-order controller. As such, the numeric controller $\widetilde{K}(z)$ is obtained through sampling with constant period $\tau \in \mathbb{R}_+$, followed by a discretization step, and using a uniform quantization $q \in \mathbb{R}_+$ of its coefficients:

$$\widetilde{K}_q = \mathcal{Q}\left\{\mathcal{D}\left\{\overline{K}, \tau\right\}, q\right\} = \mathcal{Q}\left\{\widetilde{K}, q\right\}, \tag{6}$$

while the discrete-time version of the structured singular value is:

$$\mu_{\boldsymbol{\Delta}}(\operatorname{LLFT}(\widetilde{P}, \widetilde{K}_q)) = \sup_{\omega\in\Omega_N} \frac{1}{\min\limits_{\Delta\in\boldsymbol{\Delta}}\{\overline{\sigma}(\Delta), \det(I - \operatorname{LLFT}(\widetilde{P}, \widetilde{K}_q)(e^{j\omega\tau})\Delta) = 0\}} < 1.$$

All the numerical implementation steps are performed by keeping the robustness during the entire flow, with minimum performance degradation in both transient and steady-state response. The following nonconvex optimization problem arises.

Problem 1 *Starting from a continuous-time regulator K assumed to ensure robust stability and robust performance, the implementation problem which outputs a discrete-time regulator $\widetilde{\mathcal{K}}_q \overset{D_\xi}{\sim} \mathcal{Q}\left\{\mathcal{D}\left\{K,\tau\right\},q\right\}$, given the signal quantization steps $\delta_e, \delta_x, \delta_u > 0$, with the guaranteed transient response performance and least tracking error due to quantization effects can be obtained as the solution of the constrained joint minimization problem:*

$$\min_{(\tau,q)\in\mathbb{R}^2} \min_{(\xi,\alpha)\in\mathbb{R}_+^{n_c}\times\mathbb{R}_+^{n_\ell}} \mathcal{J}\left(\xi,\alpha\right) = \varepsilon_G\left(\widetilde{\mathcal{K}}_q, \left(\overline{D}_\xi^2 D_\alpha\right) S_0, \overline{D}_\xi F\right)$$
$$s.t. \begin{cases} \left\| \left(D_\xi^{-1}\widetilde{A}_1 D_\xi, D_\xi^{-1}\widetilde{B}_1, I, O\right)\right\|_\infty < \overline{\mathcal{N}}_{x_c} \\ \mu_{\boldsymbol{\Delta}}\left(\mathrm{LLFT}\left(\widetilde{P},\widetilde{K}_q\right)\right) < 1. \end{cases} \tag{7}$$

Also, the last theoretical section of this chapter proves that the methodology also works for fractional-order interval plants, without any assumption regarding the existence of a commensurate order. This assumption removal proves the generality and versatility of the proposed method and manages to extend the problem solved in [1].

Chapter 5 aimed to present the Krasovskii passivity as an intermediate tool to extend the robust synthesis for nonlinear systems. The first major contribution is a convex manner to present the problem of studying the Krasovskii passivity of input-affine nonlinear systems. The conditions for Lemma 1 presented in **Chapter 2** have been reformulated into a finite set of linear matrix inequalities for two particular cases: bilinear case and polytopic case. As such, for an input affine nonlinear system:

$$(\Sigma): \dot{\mathbf{x}} = f(\mathbf{x},\mathbf{u}) \equiv g_0(\mathbf{x}) + \sum_{i=1}^{n_u} g_i(\mathbf{x})u_i, \tag{8}$$

with a bounded input domain $\mathcal{D}_u = [\underline{u}_1, \overline{u}_1]\times\cdots\times[\underline{u}_{n_u}, \overline{u}_{n_u}]$, if the Jacobians are constant over the state domain $\mathcal{D}_x$, i.e. the system is bilinear with $\frac{\partial g_i}{\partial \mathbf{x}} = A_i \in \mathbb{R}^{n_x\times n_x}$, $i=\overline{0,n_u}$, the following necessary and sufficient conditions for Krasovskii passivity can be given:

Theorem 1 *A bilinear system (Σ) under input bounded domain assumption is Krasovskii passive with respect to the supply-rate function $\omega_K(\mathbf{x},\mathbf{u},\mathbf{u}_d) = \mathbf{u}_d^\top h_K(\mathbf{x},\mathbf{u})$, where the output function $h_K(\mathbf{x},\mathbf{u}) = [g_1(\mathbf{x})^\top \dots g_{n_u}(\mathbf{x})^\top]\cdot Q\cdot\dot{\mathbf{x}}$ and with the storage function $S_K(\mathbf{x},\mathbf{u}) = \|\dot{\mathbf{x}}\|_Q^2$ if and only if there exists a symmetrical and positive semi-definite matrix $Q\in\mathbb{S}_{n_x}^{\geq 0}$ which satisfies the following conditions:*

$$QA_0 + A_0^\top Q + \sum_{i=1}^{n_u}\left(\left(QA_i + A_i^\top Q\right)e_i\underline{u}_i + \left(QA_i + A_i^\top Q\right)(1-e_i)\overline{u}_i\right) \leq 0, \tag{9}$$

for each binary word $\mathbf{e} = \begin{pmatrix} e_1 & e_2 & \dots & e_{n_u}\end{pmatrix}^\top \in \mathbb{Z}_2^{n_u}$.

For the polytopic systems case, i.e. $\frac{\partial g_j}{\partial \mathbf{x}} \in \mathrm{Co}\left(\mathcal{A}^{(j)} \equiv \left\{A_i^{(j)},\ i=\overline{1,n_{A^{(j)}}}\right\}\right)$, the following set of sufficient conditions has been elaborated to ensure the Krasovskii passivity.

Theorem 2 *The system (Σ) is Krasovskii passive with the supply-rate $\omega_K(\mathbf{x},\mathbf{u}) = \mathbf{u}_d^\top h_K(\mathbf{x},\mathbf{u})$, where the port variable is $h_K(\mathbf{x},\mathbf{u}) = [g_1(\mathbf{x})^\top \dots g_{n_u}(\mathbf{x})^\top]\cdot Q\cdot\dot{\mathbf{x}}$, and with the storage function $S_K(\mathbf{x}) = \frac{1}{2}\|\dot{\mathbf{x}}\|_Q^2$, if there exists a feasible matrix $Q\in\mathbb{S}_{n_x}^{\geq 0}$ that simultaneously satisfies the following conditions:*

$$Q\mathcal{A}_{\lambda^{(0)}}^{(0)} + \left(\mathcal{A}_{\lambda^{(0)}}^{(0)}\right)^\top Q + \sum_{i=1}^{n_u}\left(Q\mathcal{A}_{\lambda^{(i)}}^{(i)} + \left(\mathcal{A}_{\lambda^{(i)}}^{(i)}\right)^\top Q\right)(e_i\underline{u}_i + (1-e_i)\overline{u}_i) \leq 0, \tag{10}$$

for each binary word $\mathbf{e} = \begin{pmatrix} e_1 & e_2 & \dots & e_{n_u}\end{pmatrix}^\top \in\mathbb{Z}_2^{n_u}$ and for each singular binary word $\lambda^{(i)}\in\mathbb{Z}_2^{n_{A^{(i)}}}$.

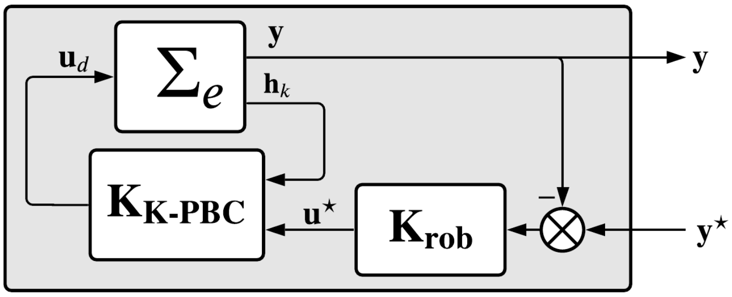

Figure 2: The proposed closed-loop system contains an inner loop having a K-PBC which ensures asymptotic stability, and an outer loop having a robust controller as a dynamical path planning for robust performance.

The conservativism of the conditions from the more general polytopic case depends on the quality of the polytopic approximation of the involved jacobians.

Another major contribution of this chapter is to propose a set of non-convex and convex mechanisms to impose (and to further minimize) the transient time of a Krasovskii passive bilinear system using first- and second-order Krasovskii passivity-based controllers. The proposed closed-loop structure is illustrated in Figure 2 and contains an inner Krasovskii controller, which ensures the asymptotic stability of the nonlinear system, alongside an outer robust component to impose a set of performances.

The parameters of the inner controller could be optimized to minimize the transient time. Each mechanism formulated in Problems 5 – 8 successfully substitutes the feasibility constraints from Problem 4. Moreover, these mechanisms could be seen as nonlinear augmentation procedures to impose a particular performance for a fixed-structure synthesis control problem. The convex optimization problems are linear programming with linear matrix inequality constraints.

This chapter also provides a unified framework to model DC-DC converters with associated nonidealities for the components, i.e. parasitic resistances or voltage drops, as relevant class of bilinear systems. For several particular topologies, the Krasovskii passivity has been studied, along with the possibility of designing a fixed-structure Krasovskii controller and an outer robust component, performing a nonlinear robust synthesis.

To illustrate the relevance of the proposed methods, the same numerical example has been used as in the original paper [7]. Even if the Krasovskii passivity along with zero detectability

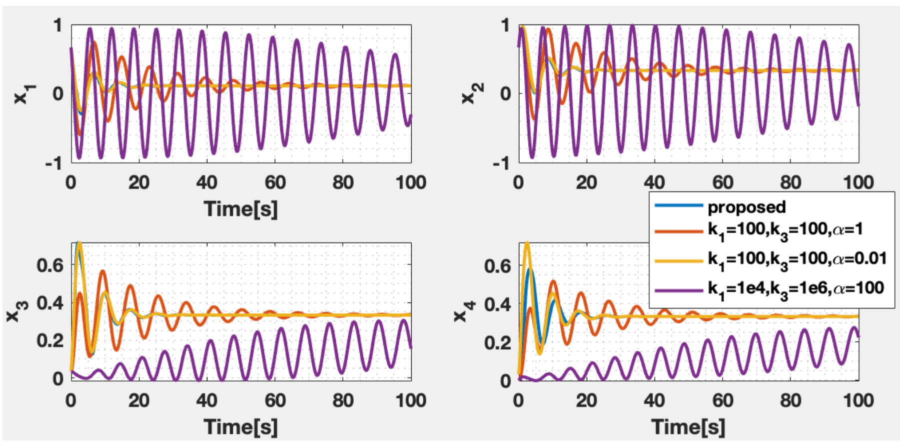

Figure 3: The evolution of the closed-loop system's state trajectories considering an initial given (possibly well calibrated) controller, along with three other combinations which ensure asymptotic stability, but with poor obtained performance.

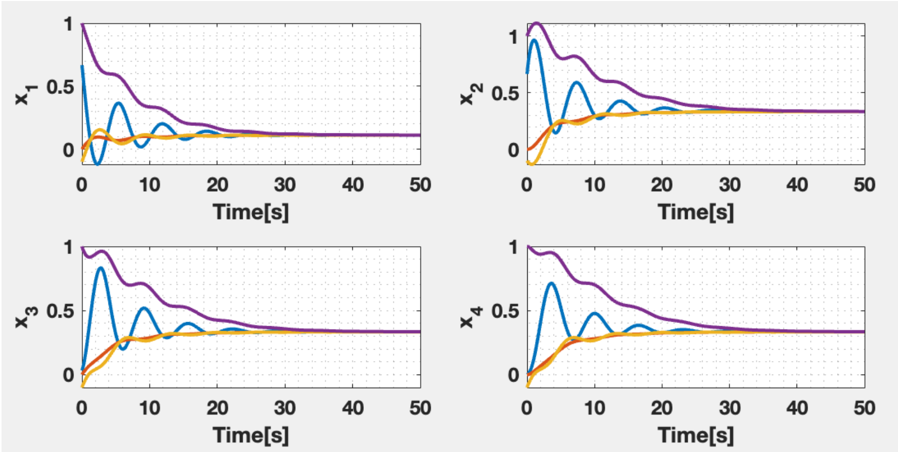

Figure 4: The closed-loop system trajectories obtained using the optimized first-order K-PBC and arbitrary initial conditions.

condition ensures the asymptotic stability, there are no available estimations of the transient time. As illustrated in Figure 3, a poor calibration of the parameters of a first-order Krasovskii passive controller implies a high value of the transient time.

If the optimized parameters are considered, a consistent transient time should be observed in Figure 4. Additionally, a comparison between the open-loop system, first- and second-order Krasovskii controllers is performed in Figure 5.

Chapter 6 extends the mathematical tools available for the *exact* feedback linearization problem to the case of *robust* feedback linearization. As stated in the literature review, the state-of-the-art solutions consider a three-layer control structure with an outer robust component designed for the uncertain linear model that encompasses the behaviour of the uncertain input-affine nonlinear system. The vast majority of the results presented in this chapter are for single-input single-output uncertain input-affine nonlinear systems, with additive uncertainties on both the input and state functions:

$$(\Sigma): \quad \begin{cases} \dot{\mathbf{x}} = f(\mathbf{x}) + \Delta f(\mathbf{x}) + (g(\mathbf{x}) + \Delta g(\mathbf{x}))\, u; \\ y = h(\mathbf{x}), \end{cases} \tag{11}$$

where the additional uncertainty maps $\Delta f, \Delta g : \mathcal{D}_{\mathbf{x}} \to \mathbb{R}^n$ are smooth in their arguments. There

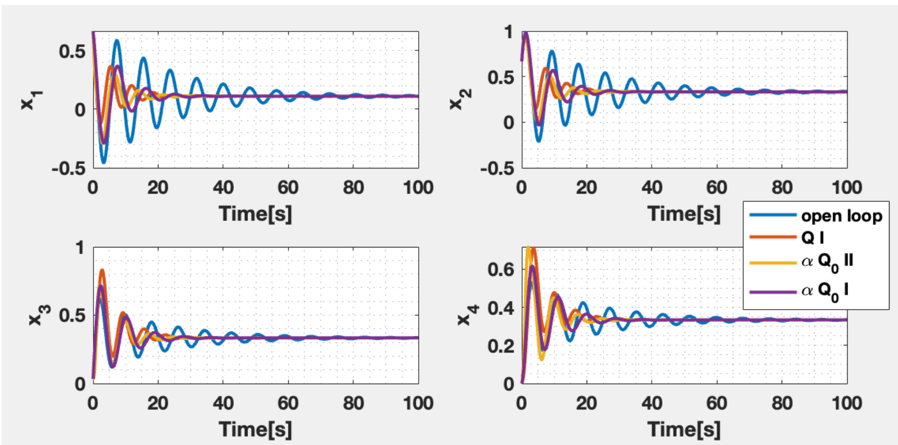

Figure 5: Comparison between the open-loop states trajectories (blue) and closed-loop trajectories obtained with a first-order K-PBC with free Q (red), and a first- (purple) and second-order (yellow) K-PBC pair with fixed Q.

are three possible classes of uncertainties: (i) fully-matched uncertainties $\Delta f, \Delta g \in \text{Span}\{g\}$; (ii) partially-matched uncertainties $\Delta g \in \text{Span}\{g\}$ and $\Delta f \notin \text{Span}\{g\}$; (iii) unmatched uncertainties $\Delta f, \Delta g \notin \text{Span}\{g\}$. One of the major contributions of this chapter is to carefully characterize the residual nonlinearities obtained after an appropriate input signal is applied to cancel the nonlinearity of the nominal system after the diffeomorphism is applied. For the general case, the state-space representation of the system (Σ) in the new coordinates $\mathbf{z} = \Phi(\mathbf{x})$ is:

$$(\Sigma_z): \begin{cases} \dot{z}_1 = z_2 + L_{\Delta f} h\left(\Phi^{-1}(\mathbf{z})\right) + L_{\Delta g} h\left(\Phi^{-1}(\mathbf{z})\right) u; \\ \dot{z}_2 = z_3 + L_{\Delta f} L_f h\left(\Phi^{-1}(\mathbf{z})\right) + L_{\Delta g} L_f h\left(\Phi^{-1}(\mathbf{z})\right) u; \\ \vdots \\ \dot{z}_{n-1} = z_n + L_{\Delta f} L_f^{n-2} h\left(\Phi^{-1}(\mathbf{z})\right) + L_{\Delta g} L_f^{n-2} h\left(\Phi^{-1}(\mathbf{z})\right) u; \\ \dot{z}_n = L_f^n h\left(\Phi^{-1}(\mathbf{z})\right) + L_{\Delta f} L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right) + L_g L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right) u + L_{\Delta g} L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right) u; \\ y = z_1. \end{cases} \tag{12}$$

Considering the linearization state feedback for the nominal system:

$$u = \frac{1}{L_g L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right)} \left(v - L_f^n h\left(\Phi^{-1}(\mathbf{z})\right)\right), \tag{13}$$

we have the following inner closed-loop system $(\Sigma_{z,\text{in}})$:

$$(\Sigma_{z,\text{in}}): \begin{cases} \dot{\mathbf{z}} = A\mathbf{z} + Bv + \widetilde{\mathbf{f}}(\mathbf{z}) + \widetilde{\mathbf{g}}(\mathbf{z})v; \\ y = C\mathbf{z}, \end{cases} \tag{14}$$

where (A, B, C) represents the chain of n integrators, while $\widetilde{\mathbf{f}}$ and $\widetilde{\mathbf{g}}$ contain n functions each, having the form:

$$\widetilde{f}_k(\mathbf{z}) = L_{\Delta f} L_f^{k-1} h\left(\Phi^{-1}(\mathbf{z})\right) - \frac{L_{\Delta g} L_f^{k-1} h\left(\Phi^{-1}(\mathbf{z})\right)}{L_g L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right)} L_f^n h\left(\Phi^{-1}(\mathbf{z})\right), \quad k = \overline{1, n}, \tag{15}$$

and:

$$\widetilde{g}_k(\mathbf{z}) = \frac{L_{\Delta g} L_f^{k-1} h\left(\Phi^{-1}(\mathbf{z})\right)}{L_g L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right)}, \; k = \overline{1, n}. \tag{16}$$

In the general unmatched uncertainty case suppose that:

- the residual state nonlinearity $\widetilde{\mathbf{f}}$ has the gradient which can be included into a convex hull of matrices $A_j \in \mathbb{R}^{n \times n}$:

$$\nabla \widetilde{\mathbf{f}} \in \text{Co}\left\{A_j, \; j = \overline{1, m}\right\}; \tag{17}$$

- the residual input nonlinearity $\widetilde{\mathbf{g}}$ is element-wise bounded:

$$\underline{\widetilde{g}_k} \leq \widetilde{g}_k(\mathbf{z}) \leq \overline{\widetilde{g}}_k, \quad \forall \mathbf{z} \in \mathcal{D}_{\mathbf{z}}. \tag{18}$$

As proved in Theorems 12, 14 and 15 from **Chapter 6**, in the case of having the uniform full relative degree, these residual nonlinearities should be encompassed by an uncertainty modeled as a descriptor system. For the general case, the theorem has the following statement:

Theorem 3 *The uncertainty which encompasses the residual nonlinearities $\widetilde{\mathbf{f}}$ and $\widetilde{\mathbf{g}}$ from $(\Sigma_{z,in})$ in the general assumption considering an inverse additive uncertainty is modeled using an improper system having the negative relative degree equal to the order of the system.*

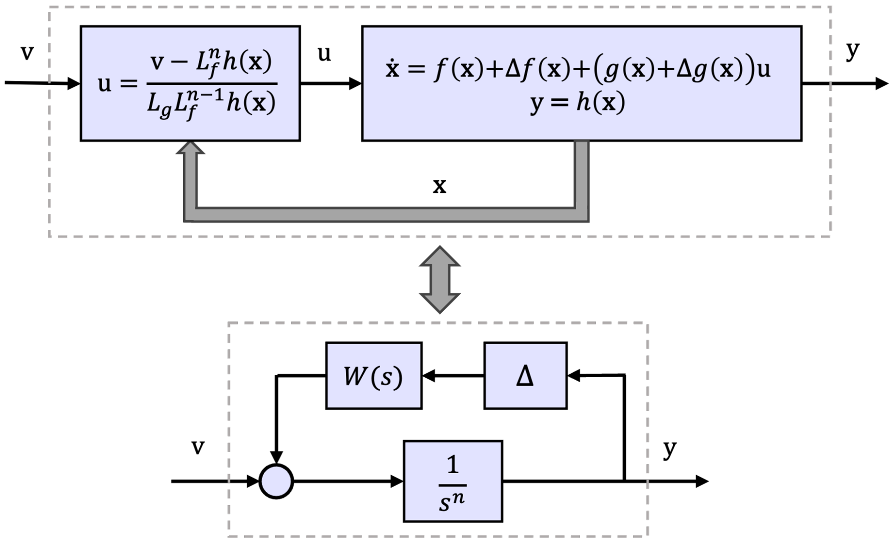

Figure 6: The equivalence between the inner closed-loop system ($\Sigma_{z,\text{in}}$) and the uncertain LTI representation $G_{\mathbf{\Delta}}(s)$.

Because the excess of zeros is equal to the order of the system, the resulting uncertain plant is proper and the robust synthesis could be successfully applied. The equivalence between the uncertain linear system and inner closed-loop system ($\Sigma_{z,\text{in}}$) is illustrated in Figure 6.

For the particular cases of partially and fully matched uncertainties, the uncertainty model $W(s)$ could be characterized in a less conservative manner. As proved in Theorem 13 from **Chapter 6**, these two types of uncertainties preserve the full relative degree of the nominal system. Additionally, for the particular case of having a uniform relative degree lower than the order of the system, the residual nonlinearities are also characterized. As such, the robust feedback linearization control problem of single-input single-output uncertain input-affine nonlinear systems can be solved by adding a single robust layer over the inner feedback linearization input.

Another major contribution is to offer a mechanism to fit an improper uncertainty model based on frequency measurements. First, the nonconvex problem has been solved using the available mechanisms. Then, a convexification procedure is presented by extending the `fitmagfrd` mechanism available in MATLAB©. A possible drawback of the method is the conservativism imposed by the hull of the uncertainty model. However, this issue is partially addressed by considering particular cases of fully and partially matched uncertainties. Moreover, the proofs of the above-mentioned theorems offer a model for the said uncertainty weighting function.

For the multi-input multi-output case, some early results are available in this chapter. The particular case of serial robots presents a uniform vector relative degree. For such uncertain nonlinear systems, the mechanism could be easily extended. However, the uncertainty modeling problem is still open for the multivariable case if the model is a descriptor system.

4 Conclusions

Part IV: contains **Chapter 7**, dealing with some final discussions and conclusions, along with a set of further research directions.

During the entire thesis, a set of theorems, lemmas, and corollaries have been stated and proved, followed by a set of proposed optimization problems. A possible drawback is the nonconvex nature of these optimization problems. However, nonconvex optimization methods can be powerful tools for solving complex optimization problems where the objective function or constraints exhibit nonconvex behavior. Even if convex optimization methods have well-established convergence guarantees, nonconvex optimization methods offer the flexibility to handle a wider range of problems, including those with non-smooth or nonconvex objective functions and constraints. The main issues of these nonconvex optimization methods are convergence to local optima, higher computational complexity, and the lack of theoretical guarantees. Overall, nonconvex optimization methods are valuable tools that complement convex optimization techniques and play a significant role in addressing control-oriented problems.

All theoretical results developed in this thesis are exposed in an end-to-end manner on relevant numerical case studies. In this context, the thesis presents another contribution in terms of the unitary treatment of DC-DC converters as bilinear systems. The case studies included in the thesis are taken and adapted from the author's publications and emphasize the evolution of the theoretical results obtained during the PhD studies. Even if the majority of these case studies are numerical simulations performed in MATLAB©/Simulink, relevant control benchmark problems have been considered.

As further research directions, there are two main problems exposed below that could be considered to solve the fixed-structure robust synthesis control problem for nonlinear systems.

Problem 2 *Let Σ be an uncertain nonlinear system of finite dimension and $\overline{\mathcal{A}}$ a nonlinear version of augmentation operator $\mathcal{A}$ which imposes a desired set of performances, inducing the nonlinear uncertain augmented plant $\Sigma_{aug} = \overline{\mathcal{A}}(\Sigma)$. Let also* CL *be the closed-loop operator between the plant Σ_{aug} and the fixed-structure controller $\Sigma_K \in \mathcal{K}_\Sigma$, where $\mathcal{K}_\Sigma$ is the equivalent of $\mathcal{K}$ for the desired family of fixed-structure nonlinear controllers. If the semi-metric $\|\|_\Sigma$ is the equivalent of the structured singular value $\mu_\Delta(\cdot)$, then the nonlinear fixed-structure robust synthesis control problem is the following optimization problem:*

$$\inf_{\Sigma_K \in \mathcal{K}_\Sigma} \| \mathrm{CL}\left(\Sigma_{aug}, \Sigma_K\right) \|_\Sigma. \tag{19}$$

Problem 3 *Let Σ be an uncertain nonlinear system of finite dimension with its nominal counterpart Σ_n. Also, let $\mathcal{L}$ be a linearization operator which transforms Σ into an uncertain LTI system $G_\Delta(s) = \mathcal{L}\left(\Sigma\right) = \mathrm{ULFT}(G_n(s), W(s)\Delta)$, where $G_n(s) = \mathcal{L}(\Sigma_n)$ is the linear representation of the nominal system, $W(s)$ is the uncertainty model, and $\|\Delta\| \leq 1$ is an arbitrary system. The augmented operator $\mathcal{A}$ induces the uncertain augmented plant $G_{aug,\Delta}(s) = \mathcal{A}(G_\Delta(s), W_s(s))$, where $W_s(s)$ contains the weighting filters used to impose the desired performances. Then the fixed-structure controller $K \in \mathcal{K}$ which ensures robust stability and performance is the solution to the optimization problem:*

$$\inf_{K \in \mathcal{K}} \mu_\Delta\left(G_{aug,\Delta}, K\right) < 1. \tag{20}$$

Problem 2 presents a set of mathematical entities which should be further defined to clearly formulate the optimization problem, considering the available optimization tools. On the other hand, Problem 3 only needs a less conservative representation of a nonlinear system in the linear control framework. Even if the Koopman operator-based linearization procedure suits the linearization operator $\mathcal{L}$, its infinite dimensionality represents an open question for the fixed-structure robust synthesis in the linear context.

References

[1] P. Apkarian, D. Noll, The $\mathcal{H}_\infty$ Control Problem is Solved, *Design and Validation of Aerospace Control Systems*, Aerospace Lab, Alain Appriou, pp. 1–11, 2017.

[2] S. Skogestad, I. Postlethwaite, *Multivariable Feedback Control: Analysis and design*, 2nd Edition, John Wiley & Sons, 2005.

[3] K.Z. Liu, Y. Yao, *Robust Control—Theory and Applications*, John Wiley & Sons, Singapore, 2016.

[4] A. Packard, J. Doyle, G. Balas, Linear Multivariable Robust Control with a μ Perspective. *J. Dyn. Syst. Meas. Ctrl.*, 115, pp. 426–438, 1993.

[5] P. Apkarian, Nonsmooth μ synthesis. *Int. J. Robust Nonlinear Control*, vol. 21, pp. 1493–1508, 2011.

[6] H.K. Khalil, *Nonlinear Control*, Global Edition, 1st Edition, Pearson Education Ltd., 2015.

[7] Y. Kawano, Y. K.C. Kosaraju, J.M.A. Scherpen, Krasovskii and Shifted Passivity-Based Control, *IEEE Transactions on Automatic Control*, vol. 66, no. 10, pp. 4926–4932, 2021.

[8] A. Isidori, *Nonlinear Control Systems*, 3rd Edition, Springer, 1995.

[9] S. Boyd, L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004.